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UNIVALENCE OF CERTAIN INTEGRAL OPERATORS

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Abstract

In this work some integral operators are studied and the authors determine conditions for the univalence of these integral operators.

1 INTRODUCTION

Let A be the class of the functions f which are analytic in the unit disc $U = \{z \in C; |z| < 1\}$ and $f(0) = f'(0) - 1 = 0$.

We denote by S the class of the function $f \in A$ which are univalent in U.

Many authors studied the problem of integral operators which preserve the class S. In this sence an important result is due to J. Pfaltzgraff [4].

THEOREM A [4]. If f is univalent in U, α a complex number and $|\alpha| \leq \frac{1}{4}$, then the function

$$G_\alpha(z) = \int_o^z [f'(\xi)]^\alpha d\xi \tag{1}$$

is univalent in U.

THEOREM B[3]. If the function $g \in S$ and α is a complex number, $|\alpha| \leq \frac{1}{4n}$, then the function defined by

$$G_{\alpha,n}(z) = \int_o^z [g'(u^n)]^\alpha du \tag{2}$$

is univalent in U for all positive integer n.

2 PRELIMINARY RESULTS

We will need the following theorems in this work.

THEOREM C [2].Let α be a complex number, $\mathrm{Re}\alpha > 0$ and $f \in A$. If

$$\frac{1-|z|^{2Re\alpha}}{Re\alpha}\left|\frac{zf''(z)}{f'(z)}\right| \leq 1 \tag{3}$$

for all $z \in U$, then for any complex number β,Re$\beta \geq$ Reα the function

$$F_\beta(z) = \left[\beta \int_o^z u^{\beta-1} f'(u) du\right]^{\frac{1}{\beta}} \tag{4}$$

is in the class S.

THEOREM D [1]. If the function g is regular in U and $|g(z)| < 1$ in U, then for all $\xi \in U$ and $z \in U$ the following inequalities hold

$$\left|\frac{g(\xi)-g(z)}{1-\overline{g(z)}g(\xi)}\right| \leq \left|\frac{\xi - z}{1-\overline{z}\xi}\right| \tag{5}$$

and

$$|g'(z)| \leq \frac{1-|g(z)|^2}{1-|z|^2}, \tag{6}$$

the equalities hold only in the case $g(z) = \epsilon\frac{z+u}{1+\overline{u}z}$ where $|\epsilon| = 1$ and $|u| < 1$.

REMARK [1]. For $z = 0$, from inequality (5)

$$\left|\frac{g(\xi)-g(0)}{1-\overline{g(0)}g(\xi)}\right| \leq |\xi| \tag{7}$$

and, hence

$$|g(\xi)| \leq \frac{|\xi| + |g(0)|}{1+|g(0)||\xi|}. \tag{8}$$

Considering $g(0) = a$ and $\xi = z$,

$$|g(z)| \leq \frac{|z| + |a|}{1+|a||z|}. \tag{9}$$

for all $z \in U$.

LEMMA SCHWARZ [1]. If the function g is regular in U, $g(0) = 0$ and $|g(z)| \leq 1$ for all $z \in U$, then the following inequalities hold

$$|g(z)| \leq |z| \tag{10}$$

for all $z \in U$, and $|g'(0)| \leq 1$, the equalities (in inequality (10) for $z \neq 0$) hold only in the case $g(z) = \epsilon z$, where $|\epsilon| = 1$.

3 MAIN RESULTS

THEOREM 1. Let α,γ be complex numbers, Re$\alpha = a > 0$ and $g \in A$.

If

$$\left|\frac{g''(z)}{g'(z)}\right| \leq \frac{1}{n} \tag{11}$$

for all $z \in U$ and

$$|\gamma| \leq \frac{n+2a}{2}\left(\frac{n+2a}{n}\right)^{\frac{n}{2a}} \tag{12}$$

then for any complex number β, $\mathrm{Re}\beta \geq a$, the function

$$G_{\beta,\gamma,n}(z) = \left\{\beta \int_0^z u^{\beta-1}\left[g'(u^n)\right]^\gamma du\right\}^{\frac{1}{\beta}} \tag{13}$$

is in the class S for all $n \in N^* - \{1\}$.

PROOF. Let us consider the function

$$f(z) = \int_0^z \left[g'(u^n)\right]^\gamma du. \tag{14}$$

The function

$$p(z) = \frac{1}{|\gamma|}\frac{f''(z)}{f'(z)}, \tag{15}$$

where the constant $|\gamma|$ satisfies the inequality (12), is regular in U.

From (15) and (14) we obtain

$$p(z) = \frac{\gamma}{|\gamma|}\left[\frac{nz^{n-1}g''(z^n)}{g'(z^n)}\right]. \tag{16}$$

Using (16) and (11) we obtain

$$|p(z)| < 1 \tag{17}$$

for all $z \in U$. For $z = 0$ we have $p(0) = 0$.

From (16) and Schwarz-Lemma it results that

$$\frac{1}{|\gamma|}\left|\frac{f''(z)}{f'(z)}\right| \leq |z|^{n-1} \leq |z| \tag{18}$$

for all $z \in U$, and hence

$$\left(\frac{1-|z|^{2a}}{a}\right)\left|\frac{zf''(z)}{f'(z)}\right| \leq |\gamma|\left(\frac{1-|z|^{2a}}{a}\right)|z|^n. \tag{19}$$

Let us consider Q:[0,1]$\rightarrow R$, $Q(x) = \frac{(1-x^{2a})}{a}x^n$, $x = |z|$. We have

$$Q(x) \leq \frac{2}{n+2a}\left(\frac{n}{n+2a}\right)^{\frac{n}{2a}} \tag{20}$$

for all $x \in [0,1]$. From (20), (19) and (12) we obtain

$$\left(\frac{1-|z|^{2a}}{a}\right)\left|\frac{zf''(z)}{f'(z)}\right| \leq 1 \tag{21}$$

for all $z \in U$. Then, from (21) and Theorem C it follows that the function $G_{\beta,\gamma,n}$, is in the class S.

THEOREM 2. Let α, γ be complex numbers, $\mathrm{Re}\alpha = b > 0$ and the function $g \in A$,
$g(z) = z + a_2 z^2 + \dots$. If

$$\left|\frac{g''(z)}{g'(z)}\right| < 1 \tag{22}$$

for all $z \in U$ and the constant $|\gamma|$ satisfies the condition

$$|\gamma| \leq \frac{1}{\max\limits_{|z|\leq 1}\left[\frac{1-|z|^{2b}}{b}|z|\frac{|z|+2|a_2|}{1+2|a_2||z|}\right]} \tag{23}$$

then for any complex number β, $\mathrm{Re}\,\beta \geq b$ the function

$$G_{\beta,\gamma}(z) = \left\{\beta \int_0^z u^{\beta-1}\left[g'(u)\right]^\gamma du\right\}^{\frac{1}{\beta}} \tag{24}$$

is in the class S.

Proof. Let us consider the function

$$f(z) = \int_0^z \left[g'(u)\right]^\gamma du. \tag{25}$$

The function

$$h(z) = \frac{1}{|\gamma|}\frac{f''(z)}{f'(z)}, \tag{26}$$

where the constant $|\gamma|$ satisfies the inequality (23), is regular in U.

From (26) and (25) we have

$$h(z) = \frac{\gamma}{|\gamma|}\frac{g''(z)}{g'(z)}. \tag{27}$$

Using (27) and (22) we obtain

$$|h(z)| < 1, \tag{28}$$

for all $z \in U$ and $|h(0)| = 2|a_2|$.

The above Remark applied to the function h gives

$$\frac{1}{|\gamma|}\left|\frac{f''(z)}{f'(z)}\right| \leq \frac{|z|+2|a_2|}{1+2|a_2||z|} \tag{29}$$

for all $z \in U$.

From (29) we obtain

$$\frac{1-|z|^{2b}}{b}\left|\frac{zf''(z)}{f'(z)}\right| \leq |\gamma|\frac{1-|z|^{2b}}{b}|z|\frac{|z|+2|a_2|}{1+2|a_2||z|} \tag{30}$$

for all $z \in U$. Hence, we have

$$\frac{1-|z|^{2b}}{b}\left|\frac{zf''(z)}{f'(z)}\right| \leq |\gamma|\max_{|z|\leq 1}\left[\frac{1-|z|^{2b}}{b}|z|\frac{|z|+2|a_2|}{1+2|a_2||z|}\right]. \tag{31}$$

From (31) and (23) we obtain

$$\frac{1-|z|^{2b}}{b}\left|\frac{zf''(z)}{f'(z)}\right| \le 1 \tag{32}$$

for all $z \in U$. From Theorem C, it follows that the function $G_{\beta,\gamma}$ defined by (24) is in the class S.

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